Completions

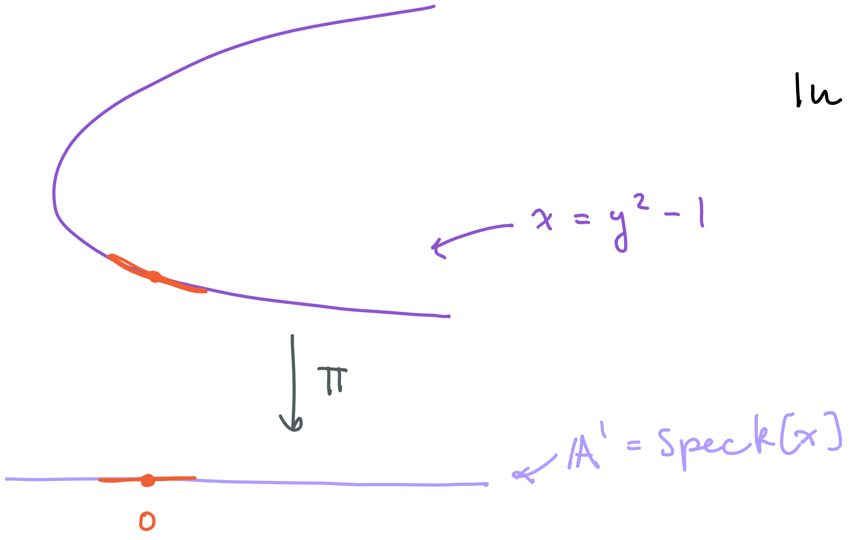
Idea: If R is a ring, $P \subseteq R$ prime, the localization tells us about Zariski open neighborhoods of P. The "completion" $\hat{R}_P$ tells us about smaller "neighborhoods". In the $\mathbb{C}$-algebra case, it tells us about neighborhoods in the "classical" topology.

We will see that if $R = k[x_1, \dots, x_n]$, $m = (x_1, \dots, x_n)$, then $\hat{R}_m = k[[x_1, \dots, x_n]]$, the formal power series ring, and

$$\widehat{\left(R/I\right)}_m = k[[x_1, \dots, x_n]] / I k[[x_1, \dots, x_n]]$$

Ex: $R = k[x,y]/(y^2 - x - 1)$. Say $k = \mathbb{R}$ or $\mathbb{C}$.

$k[x] \hookrightarrow R$ induces the map on spec:



In k^2, π is the projection onto the y-axis. In the standard topology, π has nonzero derivative at $(0,-1)$, so the Inverse function theorem says there is a neighborhood U of 0 on the line and a neighborhood V of $(0,-1)$

s.t. there is an (analytic) inverse $U \to V$, defined

$$x \longmapsto (x, -\sqrt{x+1})$$

There is no algebraic inverse since the y-coordinate would have to be a square root of $x+1$.

However, the power series expansion

$$-\sqrt{x+1} = -1 - \frac{x}{2} + \frac{x^2}{8} - \dots$$

converges for $|x|<1$, so we have an inverse at the level of power series!

We'll see that the above holds in a more general setting. First, we need the following construction.

Def: An <u>inverse system</u> is a collection of groups $\{A_i\}_{i \in J}$ with J partially ordered s.t. if $i \leq j$, $\exists$ a homomorphim $\varphi_{ij} : A_j \to A_i$ with the following properties:

1.) $\varphi_{ii} = \text{identity}$

2.) $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk} \quad \forall \; i \leq j \leq k$

i.e.

$$\begin{tikzcd} A_k \arrow[r] \arrow[d] & A_j \arrow[dl] \\ A_i & \end{tikzcd}$$

commutes.

The inverse limit of the inverse system is

$$\varprojlim A_i = \left\{ \vec{a} \in \prod_{i \in J} A_i \;\middle|\; a_j \mapsto a_i \quad \forall\, i \leq j \text{ in } J \right\}.$$

let R be a ring and $I \subseteq R$ an ideal.

Then $\{R/I^i\}_{i \in \mathbb{Z}_+}$ is an inverse system, with

$$\varphi_{ij} : R/I^j \to R/I^i$$

Def: The completion of R w.r.t. I is $\hat{R}_I$, defined

$$\hat{R}_I := \varprojlim R/I = \left\{ g = (g_1, g_2, \ldots) \in \prod_i R/I^i \;\middle|\; g_j \equiv g_i (\mathrm{mod}\, I^i),\ j > i \right\}$$

$\hat{R}_I$ is a ring with coordinate-wise addition and multiplication.

For each i, define

$$\hat{I}_i := \left\{ g = (g_1, g_2, \ldots) \in \hat{R}_I \;\middle|\; g_j = 0 \text{ for } j \leq i \right\}.$$

In $\hat{R}$, two elements are equivalent mod $\hat{I}_i$, say

$$(f_1, f_2, \ldots) \equiv (g_1, g_2, \ldots) \bmod \hat{I}_i$$

$\iff f_j = g_j \quad \forall\ j \leq i. \iff f_i = g_i.$

That is, we have a natural isomorphism

$\hat{R}/\hat{I}_i \cong R/I^i$, which is just the projection onto the ith coordinate.

If $m \subseteq R$ is maximal, then $\hat{R}_m/\hat{m}_1 \cong R/m$, a field, so $\hat{m}_1$ is maximal.

Moreover, if $g_i = (g_1, g_2, \dots) \in \hat{R}_m$, but not in $\hat{m}_1$, then $g_1 \neq 0$. Thus, each $g_i \notin m/m^i \subseteq R/m^i$.

m/m^i is the only maximal ideal of m, so g_i is a unit. Since $g_j \equiv g_i \ (\text{mod } m^i)$, it follows that $g_j^{-1} = g_i^{-1} \ (\text{mod } m^i)$, so

$$h = (g_1^{-1}, g_2^{-1}, \dots) \in \hat{R}_m$$

is the inverse of g, so g is a unit. Thus, $\hat{R}_m$ is local, w/ max'l ideal $\hat{m}_1$.

Note: $R/m^i = (R/m^i)_m = R_m/m^i_m$, so we get the same completion if we first localize at m.

Ex: $R = S[x_1, \dots, x_n]$, $m = (x_1, \dots, x_n)$. We want to show that $\hat{R}_m \cong S[[x_1, \dots, x_n]]$.

Note that $S[[x_1, \dots, x_n]] / m^i S[[x_1, \dots, x_n]] = R/m^i$

So we have a natural map

$$S[[x_1, \dots, x_n]] \longrightarrow \hat{R}_m$$
$$f \longmapsto (f + m, f + m^2, \dots).$$

In the other direction, if $(f_1 + m, f_2 + m^2, \dots) \in \hat{R}_m$, where for $i > j$, $f_i - f_j$ = terms of degree $\geq j$, then send

$$(f_1 + m, f_2 + m^2, \dots) \longmapsto f_1 + \underset{\substack{\uparrow \\ \text{terms of} \\ \text{deg } \geq 1}}{(f_2 - f_1)} + \underset{\substack{\uparrow \\ \text{terms of} \\ \text{deg } \geq 2}}{(f_3 - f_2)} + \dots$$

Thus, the coefficient of each monomial is a finite sum, so this is a formal power series. It's straightforward to check that the map is well-defined (i.e. independent of choices of f_i), and is the inverse of the above map, so

$$\hat{R}_m = S[[x_1, \dots, x_n]].$$

e.g. $(1, 1 + 2x, 1 + 2x - 3x^2, 1 + 2x - 3x^2 + x^3, \dots)$

$$\longmapsto 1 + 2x - 3x^2 + x^3 + \dots$$

Another standard example comes from number theory:

Ex: Let $p \in \mathbb{Z}$ be prime. The ring $\hat{\mathbb{Z}}_{(p)}$, written $\mathbb{Z}_p$, is the ring of p-adic integers.

Let $(a_1 + (p), a_2 + (p^2), \dots) \in \mathbb{Z}_p$ where $0 \leq a_i < p^i$.

Then for each i, $a_{i+1} \equiv a_i \pmod{p^i}$, so

$$a^{i+1} - a^i = b_i p^i, \quad b_i < p$$

and we write this as a power series, called a *p-adic expansion*:

$$a_1 + b_1 p + b_2 p^2 + \dots,$$

so that the partial sums give the sequence:

a_1

$a_1 + b_1 p = a_1 + (a_2 - a_1) = a_2$

$a_2 + b_2 p^2 = a_2 + (a_3 - a_2) = a_3$, etc.

$\mathbb{Z}/(p^i)$ has torsion, so addition of power series works a little differently. For example, in $\mathbb{Z}_2$,

$$(\underset{\substack{\uparrow\\ \text{mod } 2}}{1}, \underset{\substack{\uparrow\\ 4}}{1}, \underset{\substack{\uparrow\\ 8}}{1}, \underset{\substack{\uparrow\\ 16}}{9}, \underset{\substack{\uparrow\\ 32}}{9}, \dots) + (1, 1, 1, \dots) = (0, 2, 2, 10, 10, \dots)$$

and the corresponding power series expansions are

$$(1 + 0 \cdot 2^2 + 0 \cdot 2^4 + 1 \cdot 2^3) + (1) = (0 + 1 \cdot 2 + 0 \cdot 2^4 + 1 \cdot 2^3)$$

so addition is not term by term! Instead, we have to "carry."

e.g. in $\mathbb{Z}_3$, we have

$$(1+2\cdot 3+2\cdot 3^2)+(1+2\cdot 3+1\cdot 3^2)=2+4\cdot 3+3\cdot 3^2$$
$$=2+1\cdot 3+1\cdot 3^2+1\cdot 3^3.$$

Note that $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$ naturally, since for $r>0$, $r<p^a$ for some a, so $r \pmod{p^a} \neq 0$. i.e. the kernel is 0.

e.g. in $\mathbb{Z}_2$, $1=(1,1,1,\ldots)$, and the element $(1, 2^2-1, 2^3-1, 2^4-1, \ldots)$ has corresponding power series $1+2+2^2+\ldots$

$(1,1,1,\ldots)+(1,2^2-1,2^3-1,\ldots)=0$, so $1+2+4+\ldots=-1$.

Note that $\mathbb{Z} \not\cong \mathbb{Z}_p$: Any p-adic expansion $a_0+a_1p+a_2p^2+\ldots$ w/ $0 \le a_i < p$ corresponds to the element

$$(a_0,\ a_0+a_1p,\ a_0+a_1p+a_2p^2, \ldots) \in \mathbb{Z}_p,$$

so this gives a bijection between p-adic expansions and $\mathbb{Z}_p$. In particular, $\mathbb{Z}_p$ is uncountable!

Properties of completion

Def: If R is a ring, $I \subseteq R$ an ideal, then if the natural map $R \to \hat{R}_I$ is an isomorphism, we say R is complete with respect to I. When I is maximal, we say R is a complete local ring.

Note: $\bigcap_j I^j \longrightarrow 0$ in $\hat{R}_I$, so if R is complete w.r.t. I, then $\bigcap_j I^j = 0$.

Let $I \subseteq R$ an ideal, and denote $\hat{R} = \hat{R}_I$. We have a natural map

$$\hat{R} \to R/I^n$$
$$(f_1, f_2, \dots) \longmapsto f_n$$

Then $\hat{I}_n$ is the kernel.

Note: The elements of $I^n \hat{R}$ are generated by elements of the form $(ar_1, ar_2, \dots)$ where $a \in I^n$ and $(r_1, \dots) \in \hat{R}$.

In particular, $ar_i \in I^n$, so it's 0 for $i \leq n$.

$\Rightarrow I^n \hat{R} \subseteq \hat{I}_n$. (This is an equality if R is Noetherian, but in general, they may be different.)

However, we can always say the following about $\hat{R}$:

Claim: $\hat{R}$ is complete w.r.t. the filtration $\hat{I}_1 \supseteq \hat{I}_2 \supseteq \dots$.

That is, $\hat{R} = \varprojlim \hat{R}/\hat{I}_n$.

Pf: $\hat{R} = \varprojlim R/I^n = \varprojlim \hat{R}/\hat{I}^n$. $\square$

In the Noetherian case, we get the following:

Thm: Let R be Noetherian, $I \subseteq R$ an ideal, and $\hat{R}$ the completion w.r.t. I.

a.) $\hat{R}$ is complete w.r.t. $I\hat{R}$.

b.) $\hat{R}$ is Noetherian.

c.) $\hat{R}$ is a flat R-module.

Pf: See Eisenbud or A-M.

Cauchy sequences

Note that in the cases we've looked at, $\hat{R}$ can be thought of as "limits" of sequences in R:

Ex: In $R[x]$, the sequence $a_0, a_0 + a_1 x, a_0 + a_1 x + a_2 x^2, \dots$ "converges" to $\sum a_i x^i \in R[[x]] = \hat{R}$.

Ex: In $\mathbb{Z}_2$, $\underset{1}{\overset{\shortparallel}{1}}, \underset{3}{\overset{\shortparallel}{1+2}}, \underset{7}{\overset{\shortparallel}{1+2+2^2}}, \dots$ converges to

$1 + 2 + 2^2 + \dots = -1$.

This motivates a different characterization of completion.

Let R be a ring, I an ideal, $R^{\mathbb{N}}$ the ring of sequnces $(r_n)_{n \in \mathbb{N}}$ in R.

Def: $(r_n) \in R^{\mathbb{N}}$ is Cauchy for the I-adic topology if $\forall$ $t \in \mathbb{N}$, there is $d \in \mathbb{N}$ s.t. when $m, n \geq d$, we have $r_n - r_m \in I^t$. It converges to 0 if $\forall$ n, $\exists$ m s.t. for all $i \geq m$, $r_i \in I^n$.

Two Cauchy sequnces $(r_n), (s_n)$ are equivalent if $(r_n - s_n)$ converges to 0.

Check: Equivalence classes of Cauchy sequences form a ring, we'll call it C for now.

Note that any sequence (a_n) that is the lift of an elt of $\hat{R}$ is a Cauchy sequence: If $m \geq n$, then $a_m \equiv a_n \pmod{I^n}$, so $a_m - a_n \in I^n$.

Conversely, let (r_n) be a Cauchy sequence and fix t. Then for some d, and any $m, n \geq d$, $r_n - r_m \in I^t$. Thus, $r_n \equiv r_m \mod I^t$. If (r_n) converges to 0, then $\forall$ $n >> 0$, $r_n \in I^t$. Thus, we get a map for each t

$$C \twoheadrightarrow R/I^t \quad \text{sending} \quad (r_n) \mapsto \bar{r}_m \text{ for } m >> 0.$$

Moreover, for any $t' \leq t$, the map corresponding to t' is just the composition

$$C \twoheadrightarrow R/I^t \twoheadrightarrow R/I^{t'}$$

Claim: If C is the ring of equivalence classes of Cauchy sequences, then there is a natural isomorphism

$$\varphi : C \longrightarrow \hat{R}_I$$

given by the product of the above maps.

Pf: First note that the image of a Cauchy sequence is in $\hat{R}_I$: If $(s_n) \longmapsto (\bar{a}_1, \bar{a}_2, \ldots)$, then by the above discussion, $\bar{a}_{t'}$ is the image of $\bar{a}_t$ in the quotient, for $t' \leq t$.

It is well-defined since any Cauchy sequence converging to 0 gets sent to 0.

The map is an isomorphism since, as we described, the lift of any sequence in $\hat{R}$ is a Cauchy sequence, so we have a natural inverse. $\square$